ENTROPY: LECTURE NOTES

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> I accept chaos; I'm not sure whether it accepts me.
>
> Bob Dylan

1. Integrability or Chaos?

The goal of this course is to understand a few of the ways that mathematicians quantify the idea of *chaos*.

Chaos is a fundamental concept in dynamical systems. It shows up almost everywhere we look. Really, we expect only the most easy-to-define dynamical systems to exhibit a lack of chaos, but hitch two non-chaotic dynamical systems together in some way, and chaos is likely to emerge. (Think of a pendulum versus the double pendulum.)

Definition 1.1 (Basic dynamics terminology)**.** Let X be a set, and let $\phi : X \to X$ be any mapping. (In practice, we will ask for X to have some more structure and for ϕ to preserve that structure.) We call such mappings *dynamical systems* on X.

The *m-th iterate* of ϕ, for all $m \in \mathbb{N}$, is

$$\phi^m := \phi \circ \ldots \circ \phi \quad (m \text{ times}).$$

We set the convention that ϕ^0 is the identity. If ϕ is invertible, with inverse ψ, then for each $m \in \mathbb{N}$, we define

$$\phi^{-m} := \psi \circ \ldots \circ \psi \quad (m \text{ times}).$$

Suppose that there is a natural number $m \in \mathbb{N}$ such that ϕ^m is the identity. Then we say that ϕ has *order* m. Otherwise, we say that ϕ has *infinite order*.

A map of order 2 is called an *involution*.

Given $x \in X$, the *f-orbit of x* is the sequence of points $x, f(x), f^2(x), \ldots$. We call the points in the orbit the *iterates* of x.

An *invariant subset* $S \subseteq X$ is a subset S of X such that $f(S) \subseteq S$. The restriction $f|_S$ may be viewed then as a self-map of S.

Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are *conjugate* if there is a map $h : X \to Y$ such that

$$g = h \circ f \circ h^{-1}.$$

Conjugacy is the main notion of equivalence for dynamical systems.

1.1. Refolding.

In this course, we will study what happens when we iterate a map. Maps of finite order m are dynamically uninteresting on their own. That being said, we can take two maps of finite order, even two involutions, and compose them. This can produce maps with very surprising dynamics, as our first case study will show.

Fix an integer $n \geq 3$. A *n-gon* is an n-tuple $(v_1, v_2, \ldots, v_n)$ of points in $\mathbb{R}^2$. The points v_i are the *vertices* of the n-gon, and we treat them as cyclically ordered (so the indices are considered modulo n). We place no further restrictions on the n-gons. (We allow the points v_i to coincide or be collinear, and we do not insist on convexity.) Therefore the space of n-gons is simply $(\mathbb{R}^2)^n$.

If the vertices of an n-gon are in general linear position, the *edges* of the n-gon are the lines $\overline{v_i v_{i+1}}$, for each i in the range $1 \leq i \leq n$. In addition to the edges, it will be useful for us to consider the *short diagonals* of the n-gon, defined as the lines $\overline{v_i v_{i+2}}$ for each i in the range $1 \leq i \leq n$.

Now we introduce a self-map of $(\mathbb{R}^2)^n$ called a *flip*. To define it, choose an index i. The i-th flip is defined as the map

$$s_i : (\mathbb{R}^2)^n \dashrightarrow (\mathbb{R}^2)^n,$$

$$(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n) \mapsto (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n),$$

where v_i' is the reflection of v_i across the short diagonal $\overline{v_{i-1} v_{i+1}}$.

Notice that we have used a dashed arrow in the definition of s_i. This is because if $v_{i-1} = v_{i+1}$, then there is no clear way to define the short diagonal. Thus our map s_i is only partially defined on the domain. The map s_i is a *rational map* rather than a set-theoretic function. We will ignore this technicality until the next lecture.

Flips are involutions, so they are dynamically uninteresting. But if we compose them, we can get something cool.

The *refolding map* on the space of n-gons is the map

$$f \coloneqq s_n \circ \ldots \circ s_1.$$

It flips each vertex in order. We didn't need to compose the flips in a cyclic order! Any composition could have been interesting. We compose them this way just to fix ideas.

It's not difficult to write a program that animates the polygon as it evolves via f over time. Writing illustrative computer programs is an important method in dynamics. It lets us look for patterns which would be very hard to find by hand.

Example 1.2 (Triangles)**.** On triangles, refolding is a very simple dynamical system. Given a triangle PQR, each flip replaces PQR with a congruent triangle, so f replaces PQR with a congruent triangle. That is, there is some isometry of $\mathbb{R}^2$ that carries PQR to its image by f. Now, notice that flips respect the isometries of $\mathbb{R}^2$, in the sense that if i is any isometry, then $f \circ i = i \circ f$. (Here we are allowing i to act on the space $(\mathbb{R}^2)^3$ of triangles "diagonally" via the rule

$$i(PQR) = i(P)i(Q)i(R).$$

In particular, if i is the isometry that takes PQR to $f(PQR)$, then for any $m \in \mathbb{N}$, we have

$$f^m(PQR) = f^{m-1}(i(PQR)) = i(f^{m-1}(PQR)) = \ldots = i^m(PQR).$$

Thus f acts by an isometry. Notice, however, that i depends on PQR. So there is no one isometry that f corresponds to; rather, f transforms different triangles in different ways. This makes sense, since rescaling PQR would lead to a larger motion by f.

Exercise 1.3. It appears from Figure 1 that some iterate of f transforms PQR not just by an isometry, but by a translation. Is this true?

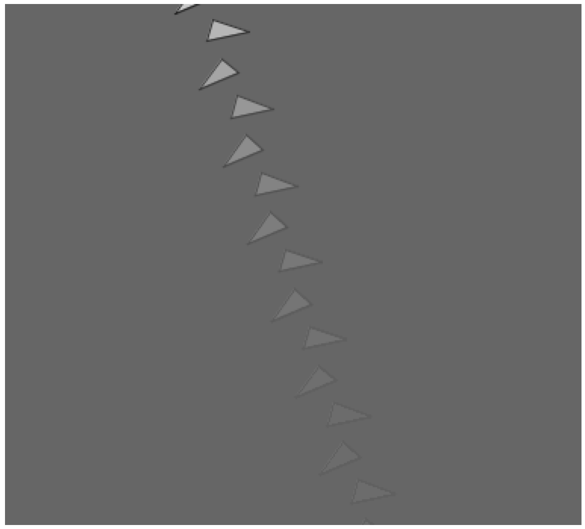

FIGURE 1. Refolding a triangle. For visual clarity we show iterates by f^{10} rather than f. The dark-to-light coloring shows the direction of time.

FIGURE 2. Refolding a quadrilateral leads to wavy motion. (Iterates by f^{10}.)

Example 1.4 (Quadrilaterals)**.** Let $n = 4$. Then $f : \mathbb{R}^8 \dashrightarrow \mathbb{R}^8$ is a self-map of the space of quadrilaterals $PQRS$. If the starting quadrilateral is very small relative to the amount of space being shown, then it will sort of look like it's waddling around the screen in a sort of "random walk." However, it turns out that this walk is very far from being random!

Even though f commutes with isometries, the argument we used for triangles can't possibly apply, since f doesn't transform $PQRS$ by an isometry. Something more complicated should happen.

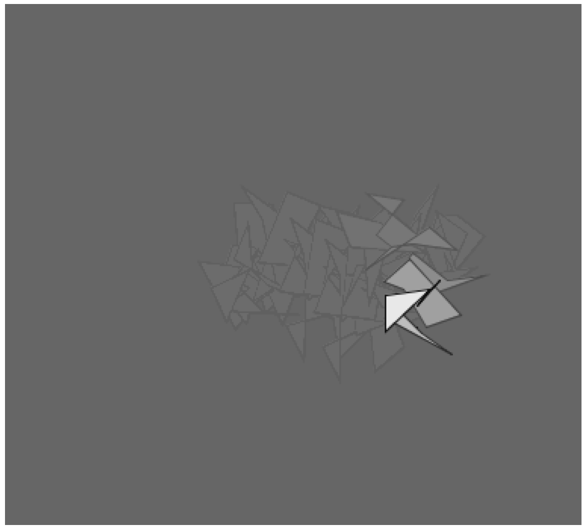

FIGURE 3. Refolding a pentagon: chaos emerges. (Iterates by f^{10}.)

Even so, the idea of "modding out by isometries" is a good one. Since f commutes with the diagonal action of the orientation-preserving isometry group

$$\mathrm{Isom}^+(\mathbb{R}^2) \cong \mathrm{SO}_2 \rtimes \mathbb{R}^2,$$

there is an induced dynamical system, denoted again by f, on the quotient space

$$(\mathbb{R}^2)^4 / \mathrm{Isom}^+(\mathbb{R}^2).$$

(We could work with orientation-reversing isometries too, but this choice of domain turns out to be very convenient.)

Proposition 1.5 (informal)**.** *For a general quadrilateral $PQRS$, the refolding orbit of $PQRS$ up to isometries can be described in a closed form using elliptic functions. In fact:*

1. *The domain $(\mathbb{R}^2)^4/\mathrm{Isom}^+(\mathbb{R}^2)$ is a union of invariant curves C_α;*
2. *A general member of the family (C_α) is the real part of an elliptic curve;*
3. *For a general member α, the map $f|_{C_\alpha}$ is algebraically conjugate to a translation map on C_α.*

Before we give the proof, let's notice that Proposition 1.5 comes quite close to giving a complete description of the refolding behavior of quadrilaterals modulo isometries. It is a machine that converts questions about dynamics into questions about (families of) elliptic curves. This is a far more detailed description than we could expect for an arbitrary dynamical system. In practice, a lot of families of dynamical systems contain some members with this kind of relationship to families of elliptic curves. For this reason, we have an informal name for dynamical systems that "look like" families of elliptic curves:

Definition 1.6 (informal)**.** A *completely integrable system* is a dynamical system that is conjugate almost everywhere to a family of translation maps on real or complex tori. By convention, we also allow maps for which some iterate has the former property, and all finite order maps are included as a degenerate case. (Points are 0-dimensional tori, for us).

Elliptic curves are complex tori, so f, viewed as a self-map of $(\mathbb{R}^2)^4/\mathrm{Isom}^+(\mathbb{R}^2)$, is a completely integrable system.

Proof of Proposition 1.5. Let the side lengths of $PQRS$ be $\ell_1, \ell_2, \ell_3, \ell_4$. The refolding map f restricts to a self-map of the space $\mathrm{Quad}(\ell)$ of quadrilaterals with the same side lengths as $PQRS$, and then descends to a self-map of the space $\mathrm{Quad}^0(\ell)$ of such quadrilaterals up to orientation-preserving isometry.

The isometry class of quadrilateral $PQRS$ may be reconstructed from the edge lengths and diagonal lengths of any representative. We give names to the squares of the edge lengths and diagonal lengths:

$$x = \overline{PR}^2, \quad y = \overline{QS}^2,$$
$$a = \overline{PQ}^2, \quad b = \overline{QR}^2, \quad c = \overline{RS}^2, \quad d = \overline{SP}^2.$$

In the real locus of $\mathrm{Quad}^0(\ell)$, the functions x and y are sufficient information to reconstruct a quadrilateral. (Over $\mathbb{R}$, the data of an edge length and its square are the same.)

However, not all values of x and y are possible. It turns out that these squared lengths obey the algebraic relation

$$(1) \quad x^2y + xy^2 - (a+b+c+d)xy + (a-d)(b-c)x + (a-b)(d-c)y + (bd-ac)(b+d-a-c) = 0.$$

(This follows from the formula for the volume of a parallelopiped in $\mathbb{R}^3$. The vanishing of the volume of the parallelopiped spanned by $P-S$, $Q-S$, $R-S$ is implied by those three vectors being coplanar.)

Notice that a, b, c, d are f-invariant. Therefore, given fixed values of a, b, c, d, each flip is an involution of the curve $C \subset \mathbb{R}^2$ in the xy-plane defined by (1). A plane cubic curve is either rational or elliptic, according to the vanishing of the discriminant of the cubic. For general values of a, b, c, d, the discriminant of (1) does not vanish (a calculation). Therefore the curve C, extended to $\mathbb{P}^2_{\mathbb{C}}$, is an elliptic curve, and each flip is an involution on it.

Precisely, the flips s_1 and s_3 leave y unchanged, but change x in general; but in (1), there are only 2 values of x per value of y. Similarly the flips s_2 and s_4 change y but not x. A standard argument with the group law of the elliptic curve then shows that the compositions $s_2 \circ s_1$ and $s_4 \circ s_3$ are translations on the cubic. (In fact s_1 and s_3 agree on $\mathrm{Quad}^0(\ell)$, as do s_2 and s_4.) □

Observation 1.7. In fact, basic algebraic geometry essentially forces one of three possibilities. Either the refolding map is actually of finite order (this seems unlikely), the complex orbits are generically stuck on genus 1 curves (a.k.a elliptic curves, but without a preferred base point), or on rational curves (that is, genus 0 curves, a.k.a. curves birational to $\mathbb{P}^1$). Let's see why.

By a quick dimension count, the quotient $(\mathbb{R}^2)^4/\mathrm{Isom}^+(\mathbb{R}^2)$ is 5-dimensional. Indeed, we can move P to the origin by a translation and then rotate so that Q is on the positive x-axis, and we can do this in exactly one way (for general $PQRS$). This leaves 5 degrees of freedom: one from Q, and two from R and S each.

Now, the map f is quite special in that it preserves all the edge lengths (and therefore also their squares). This imposes four algebraic conditions on our 5-dimensional space. Further, there aren't any algebraic relations among the squared side lengths of a quadrilateral, since we can construct a real quadrilateral with desired side lengths subject to some triangle-inequality conditions (which are semialgebraic). Therefore, each space $\mathrm{Quad}^0(\ell)$ is

1-dimensional viewed over $\mathbb{C}$. The map f restricts to an automorphism on each of these curves. But the only complex algebraic curves admitting automorphisms of infinite order are those of genus 0 or 1 (standard exercise [Har13, Ex. V.5.2]).

The invariance of edge lengths was key to understanding quadrilateral refolding. Whenever a dynamical system admits an invariant function, we want to know about it.

Definition 1.8. An *invariant function* of a dynamical system $f : X \to X$ is a function g on X such that $g \circ f = g$. An *integral* of f is a nonconstant invariant function. Any dynamical system that admits an integral is sometimes called *integrable*, but we will reserve this term mostly for completely integrable systems.

Notice in Observation 1.7 that integrals were key to proving complete integrability.

Remark 1.9. The ingenious idea in the proof of Proposition 1.5 was to look at the space of squared chord lengths. If we aren't geniuses, we need a way to guess that there might be extra structure, before we start looking for magic relations like (1). How do we "see" that an orbit is stuck on an elliptic curve?

By a quick dimension count, the quotient $(\mathbb{R}^2)^4/\operatorname{Isom}^+(\mathbb{R}^2)$ is 5-dimensional. We can write down a canonical representative of a quadrilateral $PQRS$ in its Isom^+-equivalence class by applying a translation so that the average of the four points $\frac{1}{4}(P+Q+R+S)$ is at the origin, then applying the unique rotation so that the ray PQ points towards the positive x-axis. Let the new polygon be denoted $P_0Q_0R_0S_0$. To see whether the dynamics of f "modulo isometries" are chaotic, we can plot the motion of the normalized vertices P_0, Q_0, R_0, S_0 over time. These each provide a 2-dimensional projection of the orbit of the initial polygon. This allows us to "see" the orbit.

For a typical dynamical system on a 5-dimensional space, we would expect the orbits to bounce wildly around the space, so a 2-dimensional projection should look "dusty" (see e.g. the rightmost image in Figure 5).

However, for quadrilateral refolding, instead we see *curves*, as indicated by Figure 4.

As time goes on, the iterates have motion that fills out some curve in a dense way. This curve is made up of four projections of the orbit in $(\mathbb{R}^2)^4/\operatorname{Isom}^+(\mathbb{R}^2)$ from which we can reconstruct the polygon class, so the orbit in $(\mathbb{R}^2)^4/\operatorname{Isom}^+(\mathbb{R}^2)$ appears to be stuck on a 1-dimensional subset of the space. This is odd!

We can further find experimentally that, if we replace f by a certain iterate, the motion of the polygon becomes very smooth-looking (so some f^m produces only a very small change). This is called *quasi-periodic motion*. Quasi-periodic motion also suggests a lack of chaos.

Example 1.10 (Pentagon refolding [CD23])**.** Cantat-Dujardin investigate the dynamics of refolding pentagons. Modding out $(\mathbb{R}^2)^5$ by isometries gives a 7-dimensional space, and the 5 length conditions cut the space down to real dimension at most 2. Look at Figure 5. Do you see how the leftmost picture looks like a semitransparent surface? That's just like how, when you look a semitransparent surface in real life, you see a 2-dimensional projection.

They show that the pentagon refolding extends to a holomorphic automorphism of a K3 surface, and they study its dynamics. In fact, they study the random dynamics of composing flips according to any distribution. A projection of an example of one of these K3 surface may be seen in Figure 3. Later in the course, we will see how to analyze chaos algebraically for automorphisms of a large class of K3 surfaces.

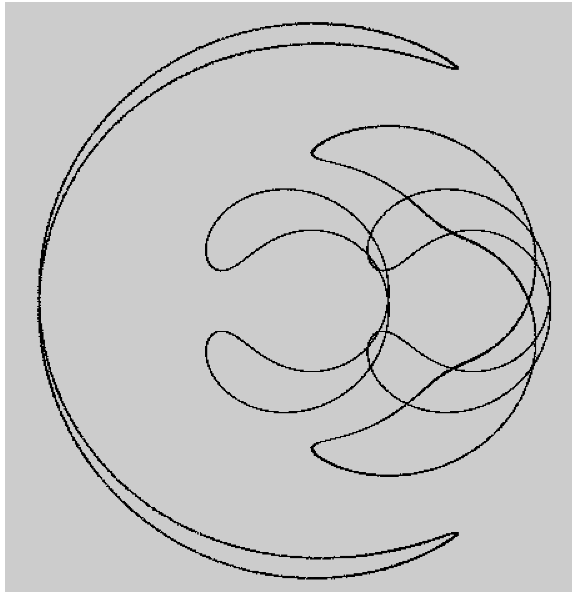

FIGURE 4. Refolding a quadrilateral: a plot of the motion of normalized vertices.

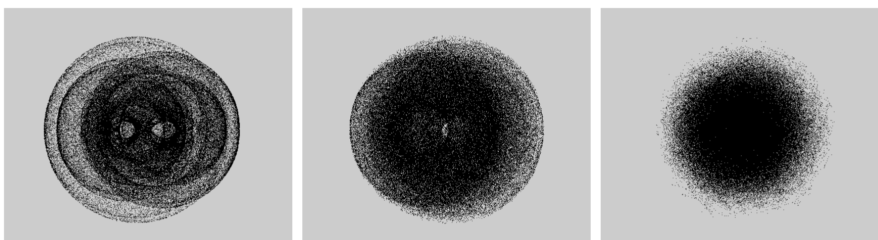

FIGURE 5. Two-dimensional projections of the refolding orbit of a 5-gon, 6-gon, and 10-gon. These orbits live on varieties of dimension 2, 3, and 7 respectively. As the number of vertices grows, the geometry becomes harder to see.

They also mention that refolding n-gons is conjugate to an automorphism of a Calabi-Yau variety of dimension $n-3$, for all $n \geq 4$. This is a generalization of the elliptic curve description of Proposition 1.5.

1.2. **The pentagram map.** Let's continue with studying interesting self-maps of polygons. As before, an n-gon is an n-tuple of points, with indices modulo n, in the real plane, with no further conditions. However, this time it will be more natural to work in the projective plane $\mathbb{P}^2_{\mathbb{R}}$.

Example 1.11 (The pentagram map)**.** Fix $n \geq 5$. The *pentagram map* on n-gons is the map

$$T : (\mathbb{P}^2_{\mathbb{R}})^n \to (\mathbb{P}^2_{\mathbb{R}})^n,$$

$$(v_1, \dots, v_n) \mapsto (v'_1, \dots, v'_n),$$

where for each index i, we define the i-th vertex of the image polygon by

$$v'_i := \overline{v_{i-1}v_{i+1}} \cap \overline{v_i v_{i+2}}.$$

Here we view the "short diagonals" as lines rather than line segments. For a general polygon, these short diagonals and the intersection v_i' exist. The degenerate polygons for which the intersections don't exist don't pose any serious difficulties in our setting.

Just as with the refolding map (Section 1.1), the pentagram map commutes with the diagonal action of a symmetry group, this time PGL_3. This is because the operations of forming diagonals and intersecting them are projectively natural. We may therefore "mod out by projectivities" just as we earlier worked modulo isometries.

Amazingly, the pentagram map on $(\mathbb{P}^2)^n/\mathrm{PGL}_3$, for *any* number n of vertices, is a completely integrable system, like refolding on quadrilaterals. Remember that the marvelous properties of quadrilateral refolding all come from the decomposition into invariant elliptic curves, which are complex tori.

Theorem 1.12. *The pentagram map T on $(\mathbb{P}^2_{\mathbb{R}})^n/\mathrm{PGL}_3$ is a completely integrable system (Definition 1.6). More specifically:*

1. *The complexified domain of the map, which has dimension $2n-8$, may be expressed as a union of T-invariant subvarieties of dimension $n-4$ or $n-5$ depending on whether n is odd or even;*
2. *Almost all of those T-invariant subvarieties are birationally isomorphic to complex tori, and T is a translation on each of them.*

Remark 1.13. The complex tori are in fact *abelian varieties*. These will show up more later, but for now we just mention that they all arise analytically as quotients of $\mathbb{C}^n$ by certain lattices, and the group laws on them are inherited from addition on $\mathbb{C}^n$.

Theorem 1.12 is proved by finding a large family of algebraically independent integrals (T-invariant functions), playing the role of the edge lengths in quadrilateral refolding. However, this time it is totally baffling what these invariant functions might be. If we could only guess them, we would be well on our way to understanding the pentagram map, but there's no clear reason to assume they're there. This leads us to a key question from mathematical physics that motivated a lot of the research that we will see in our course.

Question 1.14. How can we identify integrable systems in the wild?

Remark 1.15. There are a lot of reasons that identifying integrable systems is a natural and important problem. Completely integrable systems are exactly solvable in terms of theta functions. In physics, exactly solvable differential equations are very rare (e.g. the pendulum, or the motion of a planet around the sun) and serve as case studies for testing

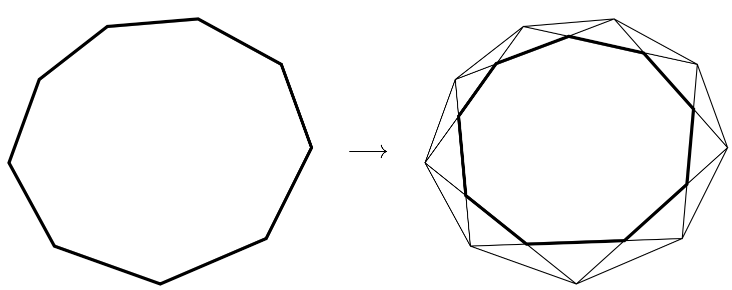

FIGURE 6. The pentagram map applied to a 9-gon. Lifted from [Wei23].

hypotheses. Exactly solvable discrete-time dynamical systems are less natural from the perspective of physics, but they have interesting relationships to differential equations that can be obtained as limits. From the dynamical systems side, integrable systems are often special members of cool families of dynamical systems. From the algebraic geometry side, completely integrable systems are related to embedding Jacobians in projective space, to rational families of abelian varieties, and to moduli spaces of vector bundles.

Definition 1.16 (Arithmetic entropy, first look)**.** Let $p/q \in \mathbb{Q}$ be a rational number written in lowest terms. The *height* of p/q is defined as

$$h(p/q) = \log \max\{|p|, |q|\}.$$

Let $x = (p_1/q_1, p_2/q_2, \dots, p_n/q_n) \in \mathbb{A}^n(\mathbb{Q})$ be a rational point. The *height* of x is

$$h(x) = \max_{1 \le i \le n}\{h(p_i/q_i)\}.$$

Let $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be a dominant rational map defined over $\mathbb{Z}$ (i.e. with only integers in the defining formulas). Let $x \in \mathbb{C}^n$ be a rational point (i.e. with all coordinates in $\mathbb{Q}$). Then we can study the sequence of heights

$$h(x), h(f(x)), h(f^2(x)), \dots$$

The *arithmetic degree of the f-orbit of x* is the limit

$$\alpha_f(x) := \lim_{m \to \infty} h(f^m(x))^{1/m}. \tag{2}$$

In general we do not know whether the limit in (2) converges, but in many cases it does. For the purposes of this chapter, we will assume that the limit converges (otherwise one can work with the lim sup instead).

The *arithmetic entropy of the f-orbit of x* is defined as

$$h_{\text{arith}}(f, x) := \log \alpha_f(x).$$

(The h is playing a dual role here, as it is the standard notation for all kinds of entropy. The arith subscript indicates that this kind of entropy is based on arithmetic information.)

Observation 1.17 (Diophantine integrability test)**.** If the arithmetic entropy $\log \alpha_f(x)$ of an orbit is positive, then the height growth in that orbit is exponential. But the arithmetic entropy of *any* translation orbit on an abelian variety is 0! This gives us an application of Diophantine geometry to dynamics and mathematical physics. Use a computer to estimate $h(f^m(x))$ for $m = 1, 2, \dots, 10$. Use the value for some large m as a guess for the arithmetic entropy. If it's very positive, then the orbit can't come from a translation, so the system cannot be completely integrable! (There's work going on under the hood here to define arithmetic entropy for arbitrary varieties. We'll cover this.)

Example 1.18 (Skew pentagram maps)**.** Now fix $a, b, c, d \in \mathbb{Z}/n\mathbb{Z}$, all distinct. Define the *skew pentagram map* with *blueprint* (a, b, c, d) to be

$$T_{a,b,c,d} : (\mathbb{P}^2_{\mathbb{R}})^n \dashrightarrow (\mathbb{P}^2_{\mathbb{R}})^n,$$
$$(v_1, \dots, v_n) \mapsto (v'_1, \dots, v'_n),$$

where

$$v'_i := \overline{v_{i+a}v_{i+b}} \cap \overline{v_{i+c}v_{i+d}}.$$

The standard pentagram map has blueprint $(-1, 1, 0, 2)$.

In [KS15], Khesin-Soloviev ask what blueprints give rise to completely integrable systems. To investigate this, they use a computer to estimate the arithmetic entropy of a randomly constructed 11-gon's orbit, for lots of different blueprints. Using Observation 1.17, they can safely conclude that positive entropy is evidence of non-integrability. Using this, they formulated a conjecture that the blueprints that give rise to complete integrability are those with

$$b - a = c - d \qquad \text{mod } n.$$

It has since been proved that these blueprints give rise to integrable systems in some reasonable sense [IK23].

However, we still lack a rigorous proof that the remaining blueprints, which are still interesting as dynamical systems, are non-integrable. Indeed the height computations for the first 10 iterates, or however many, could be misleading – perhaps the growth eventually turns out to be subexponential!

Research Project 1.19. Look into whether the skew pentagram maps are invertible (a necessary condition for complete integrability). Can this be determined from the blueprint in a straightforward way?

Remark 1.20. Historical note: the conjecture that the pentagram map is completely integrable was based on looking at projections of orbits, not height growth; see [Sch92]. Both kinds of evidence are compelling.

1.3. **Entropy.** So far, we've looked at a few dynamical systems. Some looked structured, others chaotic. One kind of "structure" that a dynamical system could have is complete integrability, and one way to gather evidence for such a structure is to use a computer to guess the arithmetic entropy of a typical orbit of that system.

However, to have a satisfying theory, we need rigorous ways of classifying dynamical systems as chaotic or not. And we really want more than this. We want to have a way of *comparing* dynamical systems quantitatively. Which is more chaotic: the skew pentagram map, or pentagon refolding? We want a number that quantifies the chaos produced by a dynamical system. This number is called *entropy*.

Of course, there are many ways to approach developing a theory of entropy. One can do dynamics of algebraic maps on varieties, smooth maps on manifolds, linear maps on vector spaces, continuous maps on metric spaces, measurable maps on probability spaces, whatever you like. We cannot reasonably ask for one notion of entropy that will work across all these categories.

What is more sensible is to look for a natural notion of entropy in each category, independently of the others. We will know that we are on the right track if there are interesting relationships (especially equalities!) between our entropies for any dynamical systems that are morphisms in multiple categories.

We're going to look at three kinds of entropy.

(1) Algebraic entropy, or dynamical degrees. These invariants are the natural way of talking about chaos in the category of algebraic varieties with rational maps. This is one of the areas of my research!
(2) Arithmetic entropy, or height growth. This is a way of quantifying the effect of a dynamical system defined over $\mathbb{Z}$ or $\mathbb{Q}$ on rational points of the domain.

(3) Topological entropy. This was historically the first notion of entropy to be developed. This notion of entropy fits well with our geometric intuition that chaos has something to do with mixing. It is the kind of entropy that is easiest to see visually, but it's delicate to define rigorously.

Besides introducing the definitions, the goal of the course is twofold. First, I want to show lots of examples of how to compute these invariants. Working out the entropy of an arbitrary dynamical system is often hopeless, so we will focus on situations where some extra algebro-geometric structure kicks in that allows us to actually compute things: algebraic tori, abelian varieties, K3 surfaces, and the projective plane.

Second, we will embark on some cross-categorical thinking. We'll see that these three flavors of entropy are related by inequalities:

$$h_{\mathrm{arith}}(f,x) \le h_{\mathrm{alg}}(f) \le h_{\mathrm{top}}(f).$$

The first inequality is due to Kawaguchi-Silverman and Matsuzawa. The second is work of Gromov and Yomdin. Both inequalities are widely useful. They allow you to transport information across categories. Cross-categorical thinking means that different parts of the course will seem easy/difficult to different students depending on background. I'm going to try to not assume too much background, but the more algebraic geometry you've seen, the better.

1.4. **Extra references and exercises.** References on refolding: it was introduced in [CR93] and related material is presented in [ER01]. Advanced material on quadrilateral refolding is in [BH04]. Pentagons are studied in [CD23].

References on the pentagram map: it was introduced in [Sch92] and complete integrability in various forms is proved in [OST10, Sol13, Wei23]. The skew pentagram maps are studied in [KS15].

Exercise 1.21. *Polygon recutting* is a dynamical system that's quite similar in spirit to polygon refolding. Let $n \ge 3$ and let $(v_1, \dots, v_n)$ be an n-gon. As before, we use cyclic indices modulo n. For each index i, the *recutting transformation* is

$$\rho_i : (\mathbb{R}^2)^n \dashrightarrow (\mathbb{R}^2)^n,$$

$$(v_1, \dots, v_n) \mapsto (v_1, \dots, v_{i-1}, v_i', v_{i+1}, \dots, v_n),$$

where v_i' is the reflection of v_i across the perpendicular bisector of the line segment $\overline{v_{i-1}v_{i+1}}$. As before, these transformations descend to maps

$$\bar{\rho}_i : (\mathbb{R}^2)^n / \mathrm{Isom}^+(\mathbb{R}^2) \dashrightarrow (\mathbb{R}^2)^n / \mathrm{Isom}^+(\mathbb{R}^2).$$

The $\bar{\rho}_i$ are involutions, so they generate a group G with respect to composition, the *recutting group.*

Let

$$g = \bar{\rho}_n \circ \dots \circ \bar{\rho}_1.$$

(1) Describe the dynamics of g on general triangles and quadrilaterals.
(2) Describe the dynamics of g on general pentagons.
(3) For more than 5, it should be difficult to guess the behavior of g. Gather computational evidence for the integrability or non-integrability of g, depending on n, and state a conjecture.

Refolding was introduced by Adler [Adl93]. It is now understood in rich detail thanks to work of Izosimov interpreting the map in terms of cluster algebras [Izo23].

Exercise 1.22. It's fun to look at examples in the literature, but it's more fun to come up with your own dynamical systems. Invent a dynamical system on $(\mathbb{R}^2)^n$ that commutes with isometries, like refolding or recutting. Modify your program from Exercise 1.21 to formulate a conjecture about the integrability or non-integrability of your dynamical system for various n.

Exercise 1.23 (Maybe hard)**.** Show that there exists a quadrilateral $PQRS$ for which the orbit by the refolding map f is infinite (non-periodic). This explains the density of the orbits on C.

Exercise 1.24 (Maybe hard)**.** Proposition 1.5 only described behavior for general quadrilaterals. Find an example of a quadrilateral $PQRS$ such that the formula for its refolding orbit is given in terms of elementary or trigonometric functions, rather than elliptic functions.

Exercise 1.25 (Pentagrammatology)**.** This exercise won't be relevant in the course, but it's fun. Prove the following basic facts about the pentagram map T.

(1) T is invertible.
(2) T is reversible, that is, birationally conjugate to its inverse.
(3) T, viewed on the quotient space $(\mathbb{P}^2_{\mathbb{R}})^n/\operatorname{PGL}_3$, is the identity when $n = 5$. (In fact T^2 is the identity when $n = 6$, but this is more annoying to prove.)

2. Rational dynamical systems

In this course, we will generally assume a background in algebraic geometry. However, this unit requires no background in algebraic geometry, so we will briefly introduce all the concepts we need. Good references are:

(1) A lighter account of dynamics of rational maps on $\mathbb{P}^n$: [Sil12, Chapter 3 and 7]
(2) Standard graduate-level introduction to regular maps and rational maps: [Har13, Chapter I]
(3) The article that this unit is closely based on: [HP07]

2.1. Rational maps on affine and projective space.

Let k be an algebraically closed field.

Definition 2.1 (Affine space)**.** For each $n \in \mathbb{N}$, affine n-space over k is denoted $\mathbb{A}^n_k$, or just $\mathbb{A}^n$ when the base field is clear from context.

We will mostly be viewing $\mathbb{A}^n$ as an algebraic variety rather than as a scheme. This means that, unless explicitly stated, the *points* of $\mathbb{A}^n$ are n-tuples $(x_1, \dots, x_n)$, where each x_i is an element of k. Thus the underlying set of $\mathbb{A}^n$ is k^n. The notation $\mathbb{A}^n$ means that we consider k^n with the Zariski topology and that the only maps we work with are algebraic ones.

Definition 2.2. A *regular map* or *morphism* of affine spaces is

$$\phi : \mathbb{A}^n \to \mathbb{A}^m,$$

$$\phi(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)),$$

where each $f_i \in k[x_1, \dots, x_n]$ for $i = 1, \dots, n$.

Regular maps are called morphisms because affine spaces together with regular maps form a category.

We will be especially interested in *endomorphisms*, which are morphisms from a space to itself, and *automorphisms*, which are invertible morphisms from a space to itself.

The endomorphisms of $\mathbb{A}^1$ are polynomials, and the automorphisms of $\mathbb{A}^1$ are linear polynomials $x \mapsto ax + b$ (where $a \neq 0$). This might lead you to think that the automorphisms of $\mathbb{A}^n$ are always "linear". However, this is far from true. For instance, it's not hard to check that

$$\phi(x, y) = (y, x + y^2)$$

admits the inverse morphism $(y, x - y^2)$. This map is nonlinear!

Definition 2.3. For an affine morphism $f : \mathbb{A}^n \to \mathbb{A}^n$, we define the *degree* $\deg f$ as the maximum degree among all the polynomials that make up the components of f. Then the degree of the affine morphism agrees with the degree of the projectivization.

Definition 2.4 (Rational maps on $\mathbb{A}^n$)**.** More generally, we want to consider *rational maps* on affine spaces. Rational maps are denoted

$$\phi : \mathbb{A}^n \dashrightarrow \mathbb{A}^m,$$

and their definition is the same except that we allow the f_i to belong to the field of rational functions $k(x_1, \ldots, x_n)$ rather than just the polynomial ring $k[x_1, \ldots, x_n]$. Remember to reduce terms (e.g. $x/(x^2) = 1/x$).

The dashed arrow $\dashrightarrow$ indicates the possible presence of an *indeterminacy locus*, i.e. the locus of points where the map is not defined. For instance, the coordinate axes form the indeterminacy locus of the map

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^2,$$

$$(x, y) \mapsto (1/x, 1/y).$$

A technical note of critical importance is that rational maps cannot always be composed. This is because the image of a rational map may be contained in the indeterminacy locus of another. This occurs, for instance, with $\phi(x, y) = (1/x, 1/(x - y))$ and $\psi(x, y) = (x, x)$.

For this reason, we will almost always require that our rational maps be *dominant*. A dominant rational map is one whose image is not contained in any subvariety of the codomain. Dominant rational maps may be composed. Hence, affine spaces and dominant rational maps form a category. That's good!

Observation 2.5. The main way in practice to check if an affine rational map $f : \mathbb{A}^n \dashrightarrow \mathbb{A}^n$ is dominant is to check that the Jacobian derivative $Df(x)$ at some point $x \in \mathbb{A}^n \setminus \operatorname{Ind} f$ is nonzero.

Definition 2.6 (Projective space)**.** For each $n \in \mathbb{N}$, projective n-space over the algebraically closed base field k is denoted $\mathbb{P}^n_k$, or usually just $\mathbb{P}^n$ when the base field is clear from context. The points of $\mathbb{P}^n_k$ are defined as the equivalence classes of vectors in $k^{n+1} \setminus \{0\}$ up to scaling by elements of $k \setminus \{0\}$. We write the equivalence class of $(X_1, \ldots, X_{n+1})$ in $\mathbb{P}^n_k$ as $[X_1 : \ldots : X_{n+1}]$. The X_i form *homogeneous coordinates* on $\mathbb{P}^n$.

There is an obvious injection of affine space $\mathbb{A}^n$ into $\mathbb{P}^n$, by sending

$$(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_n : 1].$$

This doesn't get all of $\mathbb{P}^n$ in the image because of the points with last coordinate 0. However, by changing the coordinate that is assigned value 1, we can use $n+1$ maps to cover all of $\mathbb{P}^n$ with $\mathbb{A}^n$'s, with overlaps. In this way, we can think of $\mathbb{P}^n$ as $n+1$ copies of $\mathbb{A}^n$ glued together.

The standard injection has a partially defined inverse

$$\mathbb{P}^n \dashrightarrow \mathbb{A}^n,$$

$$[X_1 : \dots X_{n+1}] \mapsto \left(\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}} \right).$$

The domain of definition is the set where $X_{n+1} \neq 0$. It is easy to check that this map is well-defined on scalar equivalence classes, thus descends to a genuine rational map on projective space.

Now we will define the fundamental objects of study in our course: rational maps of projective spaces.

Definition 2.7 (Rational maps of $\mathbb{P}^n$)**.** A *rational map* of *algebraic degree* d of projective spaces is an almost-everywhere-defined map

$$\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m,$$

$$P \mapsto [\phi_1(P) \ : \ \dots \ : \ \phi_{m+1}(P)],$$

where each $\phi_i \in k[X_1, \dots, X_{n+1}]$ is a homogeneous polynomial of the same degree $d \geq 0$, such that the ϕ_i have no common homogeneous factor. (It makes sense to speak of common factors because $k[X_1, \dots, X_{n+1}]$ is a UFD.) We often write $\deg \phi$ for the degree d.

We exclude the degenerate case where all the ϕ_i are the 0 polynomial, but the 0 polynomial is otherwise allowed to appear.

The domain of definition of ϕ is the locus

$$\{P \in \mathbb{P}^n : \text{not all the } \phi_i(P) \text{ are } 0\}.$$

We let $\operatorname{Ind} \phi$ denote the *indeterminacy locus* of ϕ, defined as the complement of the domain of definition of ϕ.

A *morphism* of projective spaces is a rational map of projective spaces for which the indeterminacy locus is empty.

As before, we will usually require our maps of projective spaces to be *dominant* (i.e. their image should be dense in the codomain). This allows us to compose and iterate.

Observation 2.8. More broadly, in algebraic geometry, two rational maps are considered equivalent if they agree pointwise on a dense subset of their domains. Removing common factors allows us to get representatives of rational map classes where the indeterminacy locus is minimal among all rational maps in that class. If we don't insist on writing our rational maps in lowest terms, then a lot of annoying things happen; in particular degree is not well-defined since e.g. $[XY : XZ : X^2] = [Y : Z : X]$.

Observation 2.9. Note that all rational maps $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ are in fact morphisms. In dimension 2 and greater, the notions become distinct. Also, the indeterminacy locus of a dominant rational map of $\mathbb{P}^n$ always has codimension at least 2, a consequence of the way rational maps are written "in lowest terms."

Definition 2.10. Another notion of degree for rational maps (affine or projective) is the *topological degree*. To define this, we recall a theorem from algebraic geometry. Given a dominant rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, there is a corresponding field extension $k(\mathbb{P}^n) \hookrightarrow k(\mathbb{P}^n)$ (it's a contravariant functor). The degree of this extension is the *topological degree* d_{top} of ϕ. It is so named because the preimage of a general point of $\mathbb{P}^m$ has cardinality d_{top}. (In finite characteristic we technically need to count these with multiplicity in order to deal with inseparable morphisms, but we won't get into this.)

Topological degree can be used to characterize birational maps:

Proposition 2.11. *If a rational map has topological degree* 1*, then it admits a rational inverse.*

This is clear because the associated field extension is degree 1, hence an isomorphism.

Remark 2.12. It turns out that the inverse of a birational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree d has degree at most d^{n-1}.

Topological degree is extremely important. In dimension 1, topological degree and algebraic degree coincide. In higher dimension, they usually differ. For instance, in $\mathbb{P}^2$, we can have birational maps of degree 2; e.g.

$$[X : Y : Z] \mapsto [YZ : XZ : XY].$$

Definition 2.13 (projectivization)**.** To come up with examples of rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$, we can *projectivize* a rational map $\mathbb{A}^n \dashrightarrow \mathbb{A}^n$. This is also called *homogenizing* a map.

To illustrate, let's consider the rational map

$$\phi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2,$$
$$(x, y) \mapsto (xy, y^{-1}).$$

We seek a map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that, as a partially-defined map of sets,

$$\Phi|_{\mathbb{A}^2} = \phi.$$

Here the restriction is that of Definition 2.6. Homogenizing, we see that the projective rational map must send

$$[x : y : 1] \mapsto [xy : y^{-1} : 1].$$

However, this formula isn't written in a way where we can see the degree (or even if it is a rational map as we defined it). We need to use homogeneous coordinates.

$$[X : Y : Z] = \left[\frac{X}{Z} : \frac{Y}{Z} : 1\right] \mapsto \left[\frac{XY}{Z^2} : \frac{Z}{Y} : 1\right].$$

This still isn't written in terms of polynomials. So we rescale again to clear denominators, making sure to do so in a way that doesn't introduce common factors:

$$\phi([X : Y : Z]) = [XY^2 : Z^3 : YZ^2].$$

Notice that there wasn't an obvious relationship between the degrees of the terms of the affine map and the degree of the projectivized map. Nonetheless, this procedure gives a 1-to-1 correspondence between dominant rational maps $\mathbb{A}^n \dashrightarrow \mathbb{A}^n$ and dominant rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$, and this correspondence respects composition.

Remark 2.14. Since dominant rational maps on $\mathbb{A}^n$ turn out to be in 1-1 correspondence with dominant rational maps on $\mathbb{P}^n$, one might wonder what the point of introducing $\mathbb{P}^n$ is. It turns out there are a lot of good reasons to do so. Intersection theory works better on $\mathbb{P}^n$. More pertinent to our course, morphisms on $\mathbb{P}^n$ are much better behaved than general morphisms on $\mathbb{A}^n$, which might have indeterminacy if viewed on $\mathbb{P}^n$.

Example 2.15. The rational map $\mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto 1/x^d$ has projectivization

$$[X : Y] \mapsto [Y^d : X^d].$$

In general, suppose a map $f : \mathbb{A}^1 \to \mathbb{A}^1$ is given by the formula $x \mapsto p(x)/q(x)$, where $p, q \in k[x]$ have no common factor. Then the degree of the projectivized map is $\max(\deg p, \deg q)$.

Exercise 2.16. Give an example of a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which cannot be iterated (i.e. $\phi \circ \phi$ cannot be defined).

Exercise 2.17. Check the claims about the 1-to-1 composition-respecting correspondence between dominant rational maps on $\mathbb{P}^n$ and $\mathbb{A}^n$.

Exercise 2.18. This exercise consists of warm-ups about understanding rational maps of projective space. Determine the indeterminacy locus of each of the following rational maps. Determine also if each of the maps is dominant. If it is non-dominant, what is the maximal subvariety of $\mathbb{P}^n$ that its image is contained in? For each of the projective maps, write the map's affine formula. For more of a challenge, try to compute the topological degrees of any dominant maps you see.

(1) The map $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ given by $f(x, y) = (y/x, x/y)$.
(2) The projectivization of f in (1), defined by $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$,

$$[X : Y : Z] \mapsto [Y^2 : X^2 : XY].$$

(3) The map $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $g([X : Y : Z]) \mapsto [X^d : Z^d : Y^d]$, for any $d \geq 1$.
(4) The map $h : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$h([X : Y : Z]) = [Y^2 - Z^2 : Z^2 - X^2 : X^2 + Y^2].$$

(5) For any $d \geq 1$, the map $j : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$j([W : X : Y : Z]) = [W^d XZ : X^d YZ : Y^d ZW : Z^d XY].$$

2.2. **The category of rational dynamical systems.**

Definition 2.19. Let $\phi : \mathbb{A}^n \dashrightarrow \mathbb{A}^n$ or $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a dominant rational map. The *m-th iterate* of ϕ, for all $m \in \mathbb{N}$, is

$$\phi^m := \phi \circ \ldots \circ \phi \quad (m \text{ times}).$$

We set the convention that ϕ^0 is the identity. If ϕ is birational, with inverse ψ, then for $m \in \mathbb{N}$ we set

$$\phi^{-m} := \psi \circ \ldots \circ \psi \quad (m \text{ times}).$$

I usually explain iteration of rational maps to non-mathematicians by remembering the filters on the Photo Booth application on Apple laptops. These filters allow you to transform an image according to some rule. Rational maps on $\mathbb{A}^2$ are a particular kind of photo filter. If you apply the filters over and over again, different kinds of things happen to the image you started with.

Any time one has an interesting category and some numbers associated to the objects (e.g. cardinality in Set, dimension in the category of varieties...) one hopes that these numbers are *invariants*, that is, that they do not change upon applying isomorphism. The natural notion of "isomorphism of dynamical systems" is birational conjugacy, but birational conjugacies can change the degree of a rational map.

Definition 2.20. Two dominant rational maps of projective space $f, g : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ are *rationally semiconjugate* if there exists a dominant rational map $h : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that

$$h \circ f = g \circ h.$$

That is, if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^n & \overset{f}{\dashrightarrow} & \mathbb{P}^n \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} \\ \mathbb{P}^n & \overset{g}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

The rational map h is a *semiconjugacy* from f to g. We say f, g are *birationally conjugate* if there is an invertible semiconjugacy h from f to g. In this situation, we have the familiar formula

$$f = h \circ g \circ h^{-1}.$$

Observation 2.21. The class of dominant rational maps on $\mathbb{P}^n$, allowing all n, form a category RatDyn. The objects are dominant rational maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. The arrows from f to g are rational semiconjugacies, and the isomorphisms in $\mathrm{RatDyn}(\mathbb{P}^n)$ are conjugacies. This means that any "intrinsic" aspects of a dynamical system are conjugacy invariant $\mathrm{RatDyn}(\mathbb{P}^n)$. Similar principles will hold in essentially any category of interest. For instance, the eigenvalues of a linear endomorphism are "intrinsic," while the matrix that realizes the map in a particular basis is not.

Example 2.22. Two rational maps of different degrees can be birationally conjugate, so the degree of a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is not a birational conjugacy invariant. This means that degree is not "intrinsic" to f; it depends on choosing a particular "model" for f. This is rather like how the notion of "degree of an algebraic curve" only makes sense once the curve is embedded in a projective space.

Consider, for instance, the linear involution

$$f([X : Y : Z]) = [Y : X : Z].$$

Conjugate f by the birational involution $h([X : Y : Z]) = [XZ : X^2 - YZ : Z^2]$. We have

$$(h \circ f \circ h^{-1})([X : Y : Z]) = [X^2Z^2 - YZ^3 : (X^2 - YZ)^2 - XZ^3 : Z^4],$$

so

$$\deg(h \circ f \circ h^{-1}) = 4,$$

It's a linear map disguised as a degree 4 map! Terrible.

For affine space, things are *even* worse than this. The issue is that degree is not really an intrinsic notion in affine space; rather, it requires a choice of embedding $\mathbb{A}^n \dashrightarrow \mathbb{P}^n$. For the sake of clarity we will always use the standard embedding.

Remark 2.23. If instead of working in the category of rational dynamical systems on $\mathbb{P}^n$, we work with regular morphisms, then degree is a well-defined notion. This is because the only regular morphisms that are available as conjugacies are linear, and these don't change the degree.

3. Dynamical degrees for $\mathbb{P}^n$

Recall that the *degree*, or *algebraic degree*, of a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is denoted $\deg f$, and the topological degree is denoted $\deg_{\text{top}} f$.

The most basic things one wants to know about a given rational self-map are usually its algebraic degree and its topological degree. As budding dynamicists, we want to understand what happens when we iterate a map. That means we want to understand the behavior of the algebraic degree and topological degree under iteration.

The story of the topological degree is easy to describe:

Lemma 3.1. *If $f, g : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ are dominant rational maps, then*

$$\deg_{\text{top}}(g \circ f) = (\deg_{\text{top}} g)(\deg_{\text{top}} f).$$

Thus, for any $N \in \mathbb{N}$, we have

$$\deg_{\text{top}} f^N = (\deg_{\text{top}} f)^N.$$

Proof. Follows quickly from the definition of $\deg_{\text{top}}$. □

The way the algebraic degree interacts with iteration, and more generally composition, is much more interesting. It's natural to expect that $\deg(g \circ f) = (\deg g)(\deg f)$, but this doesn't always happen, due to the possibility of cancelling a common factor.

Example 3.2. Consider

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2,$$
$$f([X : Y : Z]) \mapsto [YZ : XY : Z^2].$$

This is the projectivization of the affine map

$$(x, y) \mapsto (y, xy).$$

Then we may compute

$$\begin{aligned} f^2([X : Y : Z]) &= [XYZ^2 : XY^2Z : Z^4] \\ &= [XYZ : XY^2 : Z^3], \end{aligned}$$

and

$$\begin{aligned} f^3([X : Y : Z]) &= [XY^2Z^6 : X^2Y^3Z^4 : Z^9] \\ &= [XY^2Z : X^2Y^3 : Z^5], \end{aligned}$$

and

$$\begin{aligned} f^4([X : Y : Z]) &= [X^2Y^3Z^{20} : X^3Y^5Z^{17} : Z^{25}] \\ &= [X^2Y^3Z^3 : X^3Y^5 : Z^8]. \end{aligned}$$

It appears that f^N admits a closed formula in terms of Fibonacci numbers and that $\deg f^N = a_{N+2}$, where a_N is the N-th Fibonacci number. Once this formula is guessed, it's not hard to prove by induction.

When we try to describe the sequences that appear of the form $\deg(f^N)$, we find out all kinds of interesting things.

Definition 3.3. The *degree sequence* of a dominant rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the sequence of natural numbers

$$\deg f,\ \deg f^2,\ \deg f^3,\ \dots$$

The degree sequence of a map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree d is *usually* going to be

$$d, d^2, d^3, \dots$$

But we saw in Example 3.2 that other exponential growth rates are possible, since Fibonacci grows like $[(1+\sqrt{5})/2]^N$. Many other sequences can occur:

Example 3.4. The map

$$f(x,y) \mapsto \left(y, \frac{y^2+1}{x}\right)$$

has $\deg f^N = 2N$, linear growth.

Example 3.5. The map

$$f(w,x,y,z) \mapsto \left(x, y, z, \frac{xz+y^2}{w}\right)$$

has $\deg f^n \approx N^2$, quadratic growth. A weird fact about this map, and the previous one, is that all the iterates are Laurent polynomials (i.e. the denominators are all monomials). This is because seemingly magic cancellations occur that remove all non-monomial factors from the denominators as you iterate. These cancellations of course also reduce the degree of the map. This is an instance of the “Laurent phenomenon” of Fomin-Zelevinsky, related to cluster algebras [FZ02].

Observation 3.6. A natural source of rational maps in the wider world is *nonlinear recurrence sequences*. Fix some $n \in \mathbb{N}$, and consider a sequence of numbers (a_m) in some field k defined by prescribing the first n of them $a_1, \dots, a_n$. Then, for all m with $m > n$, define

$$a_m = g(a_{m-1}, \dots, a_{m-n}),$$

where g is a rational function. Any such sequence is a *recurrence sequence defined by g*. Now define

$$\phi : \mathbb{A}^n \dashrightarrow \mathbb{A}^n,$$
$$\phi(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n, g(x_{n-1}, \dots, g(x_1)).$$

Then, for each m, we have

$$a_m = x_1(\phi^m(a_n, \dots, a_1)).$$

Thus we can hope to learn about recurrence sequences by studying the associated map ϕ.

Notice that the case $n = 1$, a one-variable recurrence, encompasses all of the theory of complex dynamics of one variable, a vast subject. When $n > 1$ we should expect things to be much more complicated than when $n = 1$.

Example 3.7. Example 3.5 is the map associated to the recurrence

$$a_m = \frac{a_{m-1}a_{m-3} + a_{m-2}^2}{a_{m-4}}.$$

When this recurrence is kicked off with the initial conditions $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1$, we get a sequence of integers known as the Somos-4 sequence:

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \dots$$

This was considered very weird when it was discovered because a recurrence that uses the division operation would be expected to produce rational numbers, not integers. This is a consequence of the Laurent phenomenon described in Example 3.5.

Example 3.8. The degree of an affine morphism (Definition 2.3) has a very simple definition, but the degree growth with respect to iteration depends in very subtle ways on the map f. For instance, consider the affine morphism of degree 2 defined by

$$f(x, y) = (x, y + x^2).$$

Then

$$f^n(x, y) = (x, y + nx^2),$$

so $\deg f^n = 2$ for all n. On the other hand, consider the *Hénon map*

$$g(x, y) = (y, x + y^2).$$

There isn't a closed form for the n-th iterate f^n, but one can show that $\deg g^n = 2^n$. For a third example, consider

$$h(x, y) = (xy, y).$$

A closed form for the n-th iterate exists,

$$h^n(x, y) = (xy^n, y).$$

Thus $\deg h^n = n + 1$.

In any category, we want to have computable isomorphism invariants that we can use to tell objects apart. For instance, in the category of algebraic curves, genus is an isomorphism invariant. But in $\mathrm{RatDyn}(\mathbb{P}^n)$, degree isn't an isomorphism invariant. We want to replace "degree" with an invariant that's robust to birational conjugacy. This leads us to the first kind of entropy we examine in this course.

Definition 3.9 (Dynamical degrees + algebraic entropy)**.** Let $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a dominant rational map on $\mathbb{P}^n$. The *dynamical degree* of ϕ is the quantity

$$\delta_\phi := \lim_{N \to \infty} (\deg \phi^N)^{1/N}. \tag{3}$$

We'll prove the existence of the limit as Theorem 3.11. The *algebraic entropy* of a mapping is defined as

$$h_{\mathrm{alg}}(\phi) := \log \delta_\phi.$$

Dynamical degrees were introduced more or less simultaneously by several independent groups of mathematicians and physicists. See [RS97] for a complex dynamics angle, [BV99] for an algebraic mathematical physics angle, and the "complexity of intersections" program of Arnold [Arn90]. The terminology "algebraic entropy" is due to Bellon-Viallet.

Lemma 3.10 (Submultiplicativity)**.** *Let $f, g : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be rational maps. Then*

$$\deg(g \circ f) \le (\deg g)(\deg f).$$

Proof. In principle, we could write out a formula for $g \circ f$ and the components would be homogeneous polynomials of degree $(\deg g)(\deg f)$. After cancelling common factors (if any), we obtain a rational map of some degree $\leq (\deg g)(\deg f)$. □

It turns out that the "generic" situation is that

$$\deg(g \circ f) = (\deg g)(\deg f).$$

For instance, if g and f are morphisms, this equality holds. We'll prove this later.

To prove the existence of dynamical degrees, we want to use Lemma 3.10 to show that the degrees can't grow too quickly. We want a kind of convexity property in our sequence, but really the property we have is weaker. The best we can do is to control the sequence in arithmetic progressions, which is the idea of the proof.

Theorem 3.11. *The limit* (3) *in the definition of the dynamical degree exists.*

Proof. We prove the equivalent claim that the sequence (d_N), where $d_N = \frac{1}{N}\log\deg\phi^N$, tends to a limit as $N \to \infty$. By Lemma 3.10 applied iteratively, we always have $\deg\phi^N \leq (\deg\phi^N)$, so $0 \leq d_N \leq d_1$ for all N. It follows that the sequence (d_N) is bounded, so the sequence has a lim inf and lim sup in the interval $[0, d_1]$.

Now, let $k \in \mathbb{N}$. We claim that

$$\limsup_{N\to\infty} d_N \leq d_k. \tag{4}$$

This suffices to prove the theorem, since then taking a sequence of k's along which $d_k \to \liminf_{N\to\infty} d_N$ yields

$$\limsup_{N\to\infty} d_N \leq \liminf_{N\to\infty} d_k.$$

To prove (4), let ℓ be in the range $0 \leq \ell < k$, and notice that, by Lemma 3.10, we always have for any m that

$$\begin{aligned}\deg\phi^{mk+\ell} &\leq (\deg\phi^k)^m(\deg\phi)^\ell \\ &\leq (\deg\phi^k)^m(\deg\phi)^k.\end{aligned}$$

Taking logs,

$$(mk+\ell)d_{mk+\ell} \leq mkd_k + k\log\deg\phi.$$

It follows that

$$\begin{aligned}d_{mk+\ell} &\leq \frac{1}{mk+\ell}(mkd_k + k\log\deg\phi) \\ &\leq \frac{1}{mk}(mkd_k + k\log\deg\phi) \\ &= d_k + \frac{1}{m}\log\deg\phi\end{aligned}$$

Therefore, for each ℓ, we have

$$\limsup_{m\to\infty} d_{mk+\ell} \leq d_k.$$

Then (4) follows from observing that

$$\limsup_{N\to\infty} d_N = \max_\ell\left(\limsup_{m\to\infty} d_{mk+\ell}\right).$$

□

Theorem 3.12 (Dynamical degrees are birational invariants)**.** *Suppose that $\phi, \psi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ are dominant rational maps that are birationally conjugate; that is, there exists some birational map $g : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ such that*

$$\phi = g \circ \psi \circ g^{-1}.$$

Then $\delta_\phi = \delta_\psi$.

Proof. Easy exercise using Lemma 3.10. □

Exercise 3.13. Prove Theorem 3.12 (dynamical degrees are birational invariants).

Exercise 3.14. Show that there is no birational conjugacy between the following pair of maps $\mathbb{A}^2 \dashrightarrow \mathbb{A}^2$:

$$\phi(x, y) = (xy, y),$$
$$\psi(x, y) = (y, xy).$$

Exercise 3.15. Suppose that we're given dominant rational maps $f, g : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that, for all $N \in \mathbb{N}$, we have

$$\deg f^N = N^2, \quad \deg g^N = N^3.$$

Prove that f and g are not conjugate.

Observation 3.16. A *reversible* birational map is one that is birationally conjugate to its own inverse. As a consequence of Theorem 3.12, any reversible birational map ϕ satisfies

$$\delta_\phi = \delta_{\phi^{-1}}.$$

Exercise 3.17. It might seem in the proof of Theorem 3.11 that the sequence (d_N) is (non-strictly) decreasing. However, all that one can easily get from the submultiplicativity lemma is that (d_{2^N}) is non-strictly decreasing. Find an example of a rational map for which (d_N) is not a decreasing sequence.

Exercise 3.18. Given a dominant rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ and some $N \in \mathbb{N}$, show that

$$\delta_{\phi^N} = (\delta_\phi)^N.$$

Exercise 3.19. Suppose that $\phi, \psi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ are dominant rational maps. Prove that

$$\delta_{\phi \circ \psi} = \delta_{\psi \circ \phi}.$$

Note that this equality fails for deg.

Exercise 3.20. The *Hènon map* is the affine automorphism of $\mathbb{A}^2$ defined by

$$\phi(x, y) = (y, x + y^2).$$

Show that $\deg \phi^n = 2^n$, so $\delta_\phi = 2$.

Exercise 3.21. We have set things up so that dynamical degrees make sense over fields of finite characteristic such as $\mathbb{F}_p$. This leads to a weird and very interesting theory. Suppose that $\phi : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ is a dominant rational map that may be written (after appropriately scaling) so that all the coefficients on all the monomials in all the components of ϕ are integers, and with no simultaneous common prime factors among all these monomials. Then, for any prime p, we may ask about the dynamical system

$$\phi_p : \mathbb{P}^n_{\bar{\mathbb{F}}_p} \dashrightarrow \mathbb{P}^n_{\bar{\mathbb{F}}_p}$$

obtained by reducing the coefficients of ϕ from $\mathbb{Z}$ to $\mathbb{F}_p$. In general, this reduction procedure produces a dominant rational map on $\mathbb{P}^n_{\bar{\mathbb{F}}_p}$. Consider now the particular example with affine formula

$$\phi(x,y) = \left(2x, y\frac{3x+1}{x+1}\right).$$

Describe the degree sequence and find the dynamical degree of ϕ; in particular, show that as $n \to \infty$, we have

$$\deg \phi^n \to \infty.$$

On the other hand, show that for *every* prime p, we have

$$\deg(\phi_p)^n \not\to \infty.$$

Thus ϕ may be thought of as a map that is "chaotic over $\bar{\mathbb{Q}}$ but not over any $\bar{\mathbb{F}}_p$." This example was shown to me by Shengyuan Zhao.

Remark 3.22. Xie [Xie15, Section 5] found the following amazing example. The birational map

$$\phi : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}},$$
$$f([X:Y:Z]) = [XY : XY - 2Z^2 : YZ + 3Z^2].$$

has

$$\delta_\phi = 2,$$

but for all primes p, we have

$$\delta_{\phi_p} < 2.$$

It is proved in that article that for any $\psi : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ defined over $\mathbb{Z}$, we have

$$\lim_{p\to\infty} \delta_{\psi_p} = \delta_\psi.$$

Thus the dynamical degrees of the reductions tend to the "correct" value.

4. Dynamical degrees of monomial maps

Definition 4.1. Let $A \in \mathrm{Mat}_n(\mathbb{Z})$ be a nonsingular $n \times n$ matrix (a_{ij}). The *monomial map* associated to A is the rational map

$$\phi_A : \mathbb{A}^n \dashrightarrow \mathbb{A}^n,$$
$$(x_1, \dots, x_n) \mapsto (x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}}, \ \dots, \ x_1^{a_{n1}} x_2^{a_{n2}} \dots x_n^{a_{nn}}).$$

Example 4.2. The matrix

$$A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$$

gives

$$\phi_A(x,y) = (x^{-1}y^2, x^{-3}y^{-4}).$$

Monomial maps are often the easiest test cases when forming hypotheses about the theory of rational maps. This is because the iterates of a monomial map can be written down in (essentially) a closed form, due to the following functoriality-like statement.

Lemma 4.3. *Let $A, B \in \mathrm{Mat}_n(\mathbb{Z})$ be nonsingular. Then*
$$\phi_{BA} = \phi_B \circ \phi_A.$$
In particular, for any $N \in \mathbb{Z}$, we have
$$\phi_{A^N} = (\phi_A)^N.$$

Proof. Immediate from laws of exponents. □

Observation 4.4. A neat feature of monomial maps over $\mathbb{C}$ is that they have an invariant set, the standard real n-torus
$$\mathbb{T}^n := \{(x_1, \dots, x_n) : |x_1| = \dots = |x_n| = 1\} \subset \mathbb{C}^n.$$
Identify $(\mathbb{R}/\mathbb{Z})^n$ with $\mathbb{T}^n$ via the coordinatewise exponential map
$$(\theta_1, \dots, \theta_n) \mapsto (e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}).$$
Then in the θ_i-coordinates, the action of the set-theoretic map
$$\phi_A|_{\mathbb{T}^n} : (\mathbb{R}/\mathbb{Z})^n \to (\mathbb{R}/\mathbb{Z})^n$$
is simply
$$\begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix} \mapsto A\begin{bmatrix}\theta_1\\ \vdots\\ \theta_n\end{bmatrix}.$$
Thus the monomial map ϕ_A can be thought of as a meromorphic extension of the toral endomorphism that A defines.

One consequence of this observation is that the monomial map ϕ_A should be at least as dynamically complicated as A. However, we have no way yet of comparing the complexity of a real map and an algebraic map. We'll see this later in the course.

Exercise 4.5. In the construction of Definition 4.1, show that ϕ_A is dominant if and only if A is a nonsingular matrix. This is the reason why we required A to be nonsingular.

Exercise 4.6. Show that the topological degree of ϕ_A is equal to $|\det A|$. (Hint: elementary matrices.)

Note that if all the a_{ij} are nonnegative, then ϕ_A is an affine morphism, but otherwise there may be indeterminacy. If we restrict to the complement of the coordinate hyperplanes, then we get a bona fide regular morphism on the "algebraic n-torus." This point of view allows one to study monomial maps from the point of view of toric varieties, and many of the strongest results on monomial maps make ample use of that theory.

The projectivization Φ_A of ϕ_A is given in homogeneous coordinates $X_1, \dots, X_{n+1}$ by
$$\begin{aligned}
[X_1 : \dots : X_{n+1}] = \left[\frac{X_1}{X_{n+1}} : \dots \frac{X_n}{X_{n+1}} : 1\right] \\
\mapsto \Bigg[\left(\frac{X_1}{X_{n+1}}\right)^{a_{11}} \left(\frac{X_2}{X_{n+1}}\right)^{a_{12}} \dots \left(\frac{X_n}{X_{n+1}}\right)^{a_{1n}} \\
: \dots \\
: \left(\frac{X_1}{X_{n+1}}\right)^{a_{n1}} \left(\frac{X_2}{X_{n+1}}\right)^{a_{n2}} \dots \left(\frac{X_n}{X_{n+1}}\right)^{a_{nn}} \quad : \quad 1\Bigg].
\end{aligned}$$

Note that, even if the affine map ϕ_A has no indeterminacy, the projective map Φ_A may have indeterminacy on the "line at infinity."

From now on, we won't continue to distinguish between ϕ_A and Φ_A. They are two incarnations of the same object.

In order to see the degree of $\phi_A : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, we must write it in a denominator-free way. We can do this by multiplying every component by a suitable monomial. We need to multiply by the maximum power of each X_i appearing among all the denominators in the first n components. For each $1 \le j \le n$, the largest power of X_j in a denominator is $\max_{i=1,\dots,n}\{0, -a_{ij}\}$. The largest power of X_{N+1} in a denominator is

$$\max_{i=1,\dots,n}\{0, \sum_{j=1}^{n} a_{ij}\}.$$

This yields:

Proposition 4.7. *Let A be an $n \times n$ nonsingular matrix. The algebraic degree of the associated projective monomial map $\phi_A : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is*

$$\deg \Phi_A = \sum_{j=1}^{n} \max_{i=1,\dots,n}\{0, -a_{ij}\} + \max_{i=1,\dots,n}\{0, \sum_{j=1}^{n} a_{ij}\}.$$

x

Example 4.8. The projectivization of

$$\phi_A(x, y) = (x^{-1}y^2, x^{-3}y^{-4})$$

is the degree 8 map

$$\Phi_A([X : Y : Z]) = [X^2Y^6 : Z^8 : X^3Y^4Z].$$

The computation of the dynamical degree of a monomial map is due to Hasselblatt-Propp in 2007 [HP07, Theorem 6.2].

Theorem 4.9. *Let $A \in \mathrm{Mat}_n(\mathbb{Z})$ be a nonsingular matrix, and let $\phi_A : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the associated projective monomial map. Then the dynamical degree of ϕ_A is the spectral radius of A (the maximum absolute value among the eigenvalues of A).*

Proof. According to Lemma 4.3, we have for all $N \in \mathbb{N}$ that $(\phi_A)^N = \phi_{A^N}$. Therefore the degree sequence is obtained by evaluating the degree formula of Proposition 4.7 with the powers A^N in place of the original matrix A. Therefore the essential task is to estimate the individual entries of the powers A^N and the row sums in the powers A^N.

We treat the case where A is diagonalizable and leave the full statement of the theorem as an exercise.

Let r be the spectral radius of A. Since we are assuming that A is diagonalizable, and the case $A = \mathrm{Id}$ is obvious, we assume that $A \neq \mathrm{Id}$. The entries of A^N are each at most Nr^N; indeed the norms of the columns of A^N are at most Nr^N, which is easy to see by expanding the standard unit vectors in an eigenbasis. (Big-O means at most up to a constant.) It easily follows from the degree formula that $\deg \phi_{A^N}$ is $O(Nr^N)$, so $\delta_{\phi_A} \le r$.

To see that $\delta_{\phi_A} \ge r$, we argue as follows. Say that $\deg \phi_{A^N} \le O(c^N)$ for some value c such that $1 < c$. We claim that then $c \ge r$.

For each $\epsilon > 0$, for all $N \gg_\epsilon 0$, we have

$$\deg \phi_{A^N} < (c + \epsilon)^N.$$

By the degree formula, we have for each entry i, j in the range $1 \le i \le n, 1 \le j \le n$ that

$$(A^N)_{ij} \ge -\deg \phi_{A^N} > -(c+\epsilon)^N$$

and for each i in the range $1 \le i \le n$ we have that

$$\sum_{j=1}^{n} (A^N)_{ij} \le \deg \phi_{A^N} < (c+\epsilon)^N.$$

Combining these, the maximum value of any $(A^N)_{ij}$ is

$$n \deg \phi_{A^N} < n(c+\epsilon)^N.$$

So we have upper bounds of $n(c+\epsilon)^N$ on the absolute values of the entries of A^N.

Now, let u be a unit r-eigenvector of A. Then $\|A^N u\| = r^N$. But since the components of u are each at most 1 in absolute value, the components of $A^n u$ are each at most $n^2(c+\epsilon)^N$ in absolute value, and so $\|A^n u\| \le n^{5/2}(c+\epsilon)^N$. It follows that

$$r \le \sqrt[n]{n^{5/2}(c+\epsilon)^N}.$$

This holds for all sufficiently large N, so let $N \to \infty$; then $r \le c + \epsilon$. This works for any ϵ, so $r \le c$. □

Exercise 4.10. Generalize the argument so that it applies to non-diagonalizable matrices. Make sure your argument works for the monomial map (x, xy).

Exercise 4.11. Give examples of monomial maps with the following degree sequences, or prove that it is impossible.

- $2, 4, 8, 16, \ldots$
- $2, 3, 4, 5, \ldots$
- A quadratic growth rate (i.e. on the order of $4, 9, 16, 25, \ldots$)
- $2, 3, 5, 8, 13, \ldots$
- $2^N + N2^{N-1}$.
- 2^{2^N}.
- $\lceil \sqrt{N} \rceil$.
- $3, 1, 4, 1, 5, 9, 2, \ldots$

Exercise 4.12. Find an example of a birational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\deg \phi \ne \deg \phi^{-1}$.

Exercise 4.13. Give an example of a birational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\delta_\phi \ne \delta_{\phi^{-1}}$.

Exercise 4.14. Give an example of a pair of rational maps $f, g : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that

$$\deg(f \circ g) \ne \deg(g \circ f).$$

Exercise 4.15. A *translated monomial map* is a map of the form $q \circ \phi_A$, where ϕ_A is a monomial map and $q : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a map of the form

$$q([X_1 : \ldots : X_{N+1}]) = [q_1 X_1 : \ldots : q_{N+1} X_{N+1}].$$

Compute the dynamical degree of a translated monomial map.

Exercise 4.16. We showed that the dynamical degree is a birational invariant. The topological degree is also a birational invariant. Find a pair of monomial maps that have the same dynamical degree and topological degree, but are not birationally conjugate.

Exercise 4.17. Find a birational monomial map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree d with inverse of degree d^{n-1}.

Research Project 4.18. The following is a possibly open research question. The monomial maps of a fixed algebraic degree d form a finite set of rational maps of $\mathbb{P}^n$. It shouldn't be hard to determine the cardinality of this set. However, from the perspective of dynamics, many of these maps are in fact the same (they are birational conjugates). For instance, the matrices A and BAB^{-1} for any $B \in \mathrm{GL}_n(\mathbb{Z})$ give rise to birationally conjugate dynamical systems. How many distinct classes up to $\mathrm{GL}_n(\mathbb{Z})$-conjugacy are there in $\mathrm{Mat}_n(\mathbb{Z})$ up to a given algebraic degree? (Over $\mathbb{Q}$ this would be a problem about Jordan normal form, but we're talking about $\mathbb{Z}$; this distinction is apparently related to ideal class groups.) Is it possible for monomial maps to be birationally conjugate via a non-monomial map? What is the group of birational self-conjugacies of a given monomial map? Can you estimate the number of birational conjugacy classes that are represented by monomial maps up to algebraic degree d?

Exercise 4.19. Given n and d, find examples of rational maps of $\mathbb{P}^n$ of degree d and topological degree $1, 2, \ldots, d$. Also find examples with topological degree $d, d^2, \ldots, d^n$.

Research Project 4.20. Classify the possible topological degrees for a monomial map of degree d.

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